Taylor – Couette vortices in a yield – stress fluid in presence of a static layer

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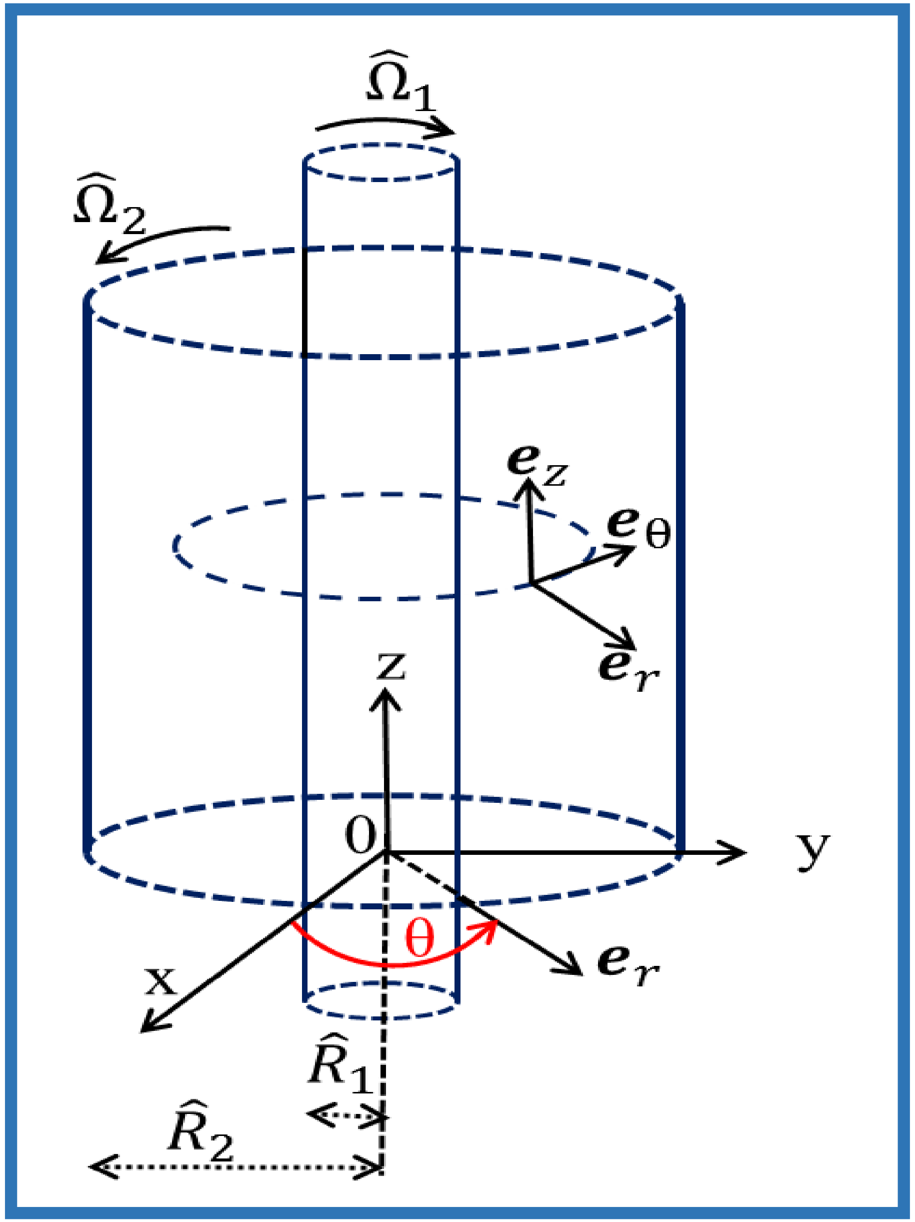
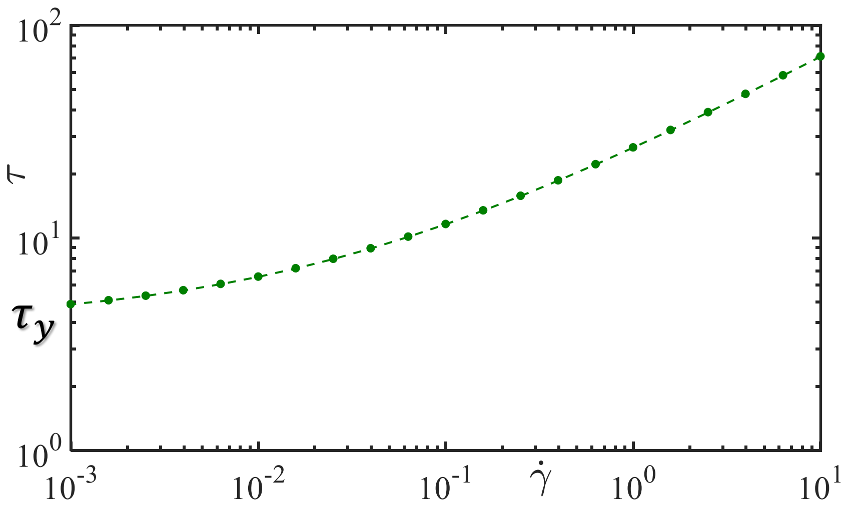
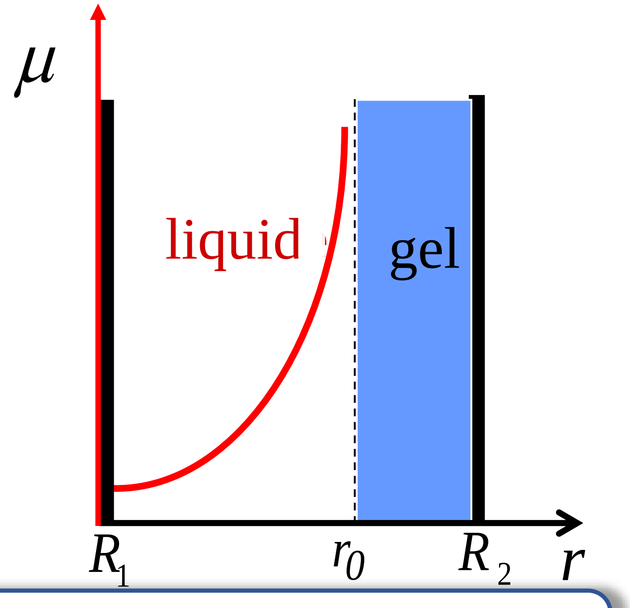
Description of the problem



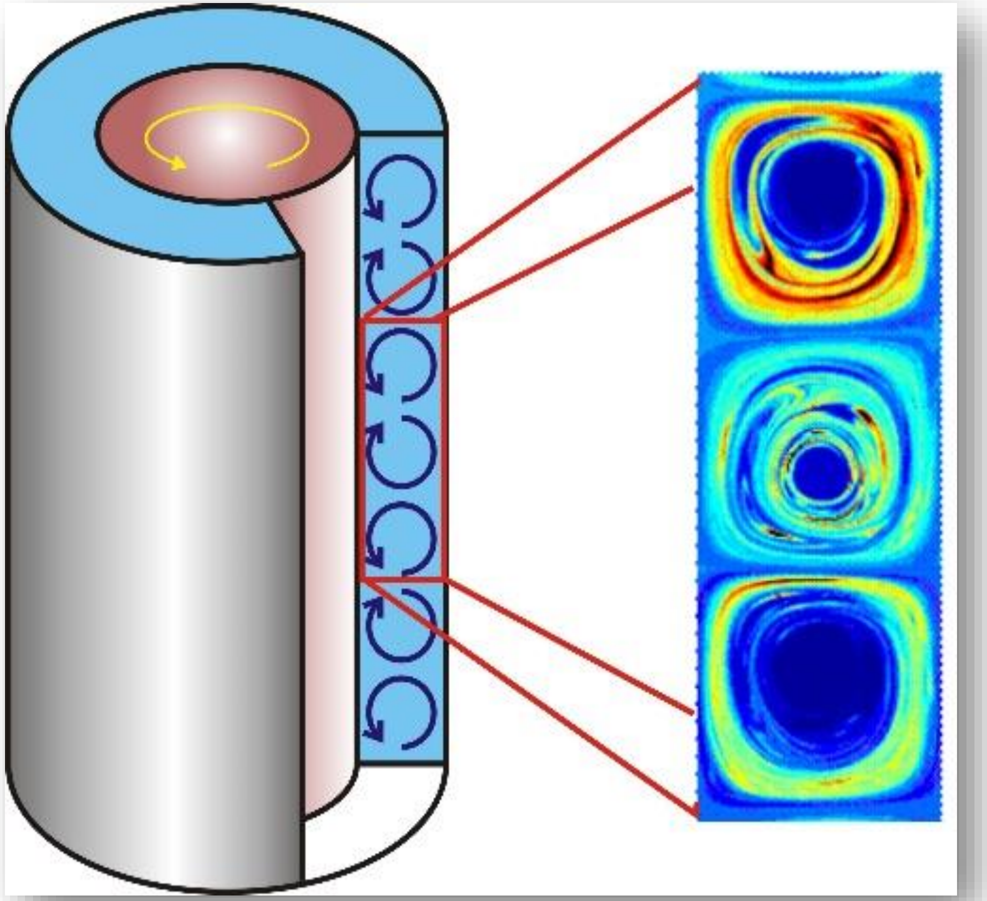
Flow of a yield – stress fluid between two infinite coaxial cylinders rotating with angular velocities Ω_1, Ω_2

Yield stress fluid: behaves as a liquid, when the shear-stress is higher than the yield-stress τ_y and as a “solid - like” when the shear-stress is less than τ_y





Viscosity tends to infinity at the yield surface



Same physical mechanism for non-Newtonian purely viscous fluids

Taylor – Couette flow in a yield – stress fluid between two infinite coaxial cylinders.

According to the Rayleigh criterion, when unstable stratification of the angular momentum :

$$\frac{d(rV_b)^2}{dr} < 0 \ , \forall\, r = [R_1, R_2]$$

flow becomes unstable, which leads to the appearance of Taylor vortices. Viscosity has the role of dumping this effect, otherwise, without viscosity vortices appear immediately.

❖ Objective:

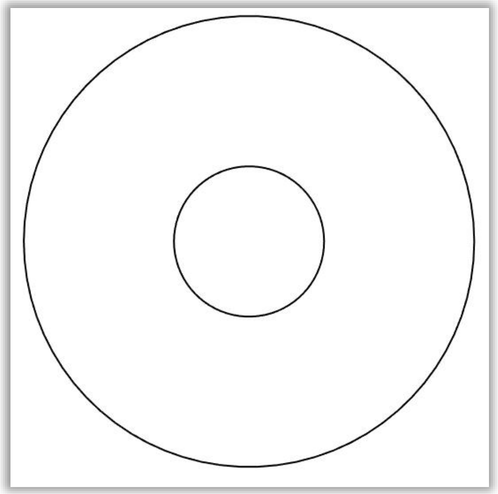
Understand the flow structure and the behavior of the Taylor vortices in a yield-stress fluid, particularly in the case where we have a static layer. Two situations are considered: wide and narrow gaps.

❖ Methodology:

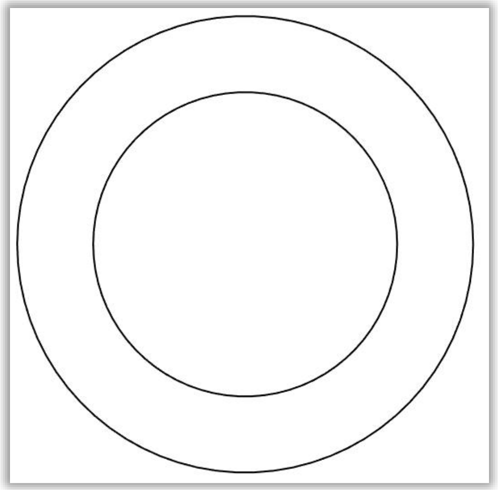
Linear and weakly nonlinear stability analysis

❖ Difficulty:

Difficulty comes from the fact that at the yield surface the viscosity tends to infinity. It will be explained during analysis.



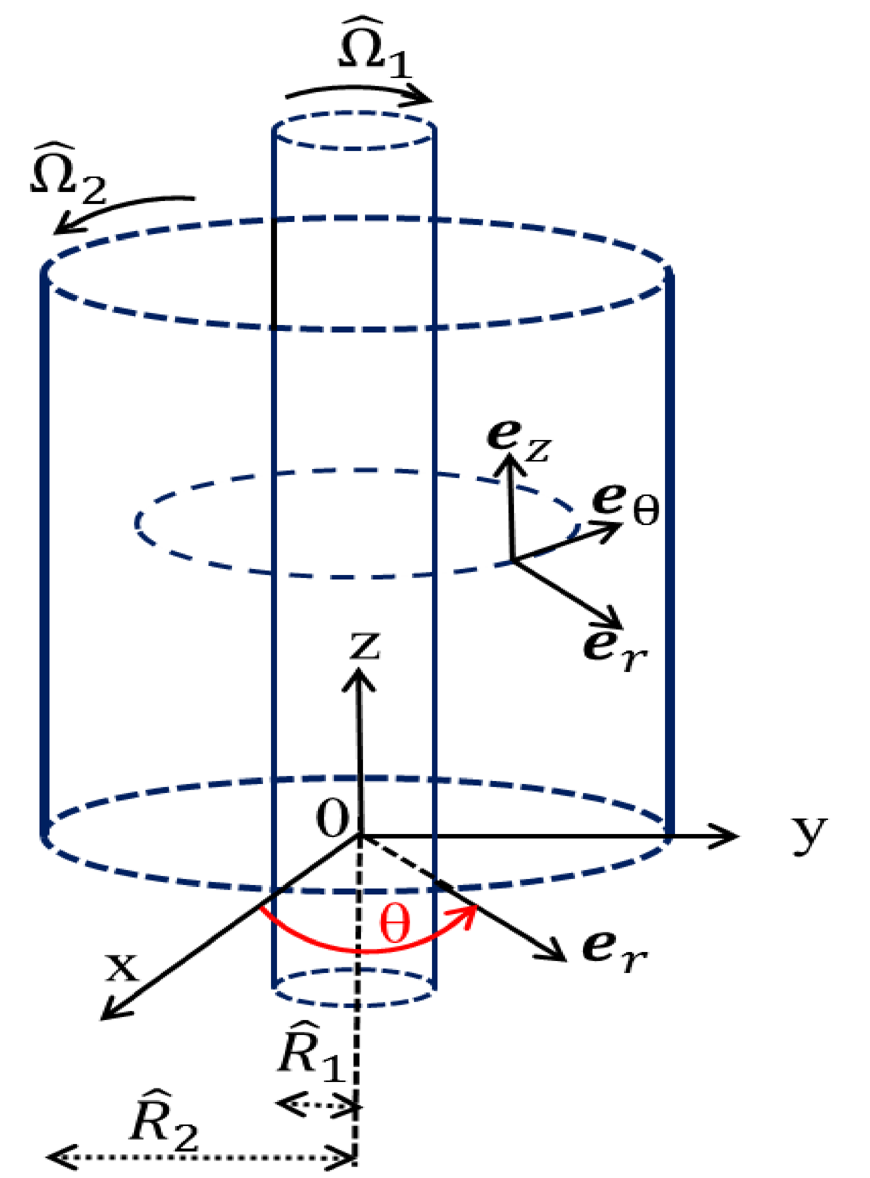
$\eta = 0.4$
Wide gap



$\eta = 0.883$
Narrow gap

$$\eta = \frac{R_1}{R_2}$$

General equations



$$\boldsymbol{\nabla} \cdot \boldsymbol{U} = 0$$

$$\frac{\partial \boldsymbol{U}}{\partial t} + Re_1 \left(\boldsymbol{U} \cdot \boldsymbol{\nabla}\right)\boldsymbol{U} = -\boldsymbol{\nabla} P + \boldsymbol{\nabla} \cdot \boldsymbol{\tau}$$

Characteristic scales:

- Length : $\hat{d} = \hat{R}_2 - \hat{R}_1$
- Velocity : $\hat{\Omega}_1 \hat{R}_1$
- Reynolds number of inner cylinder: $Re_1 = \frac{\hat{\rho}\hat{\Omega}_1\hat{R}_1\hat{d}}{\hat{\mu}_{ref}}$
- Reynolds number of outer cylinder: $Re_2 = \frac{\hat{\rho}\hat{\Omega}_2\hat{R}_2\hat{d}}{\hat{\mu}_{ref}}$

Bingham model

Constitutive equation

- **Dimensional form**

$$\hat{\tau} = \left[\hat{\mu}_p + \frac{\hat{\tau}_y}{\dot{\hat{\gamma}}}\right]\dot{\hat{\gamma}} \quad \Longleftrightarrow \quad \hat{\tau} > \hat{\tau}_y$$

$$\dot{\hat{\gamma}} = 0 \quad \Longleftrightarrow \quad \hat{\tau} \le \hat{\tau}_y$$

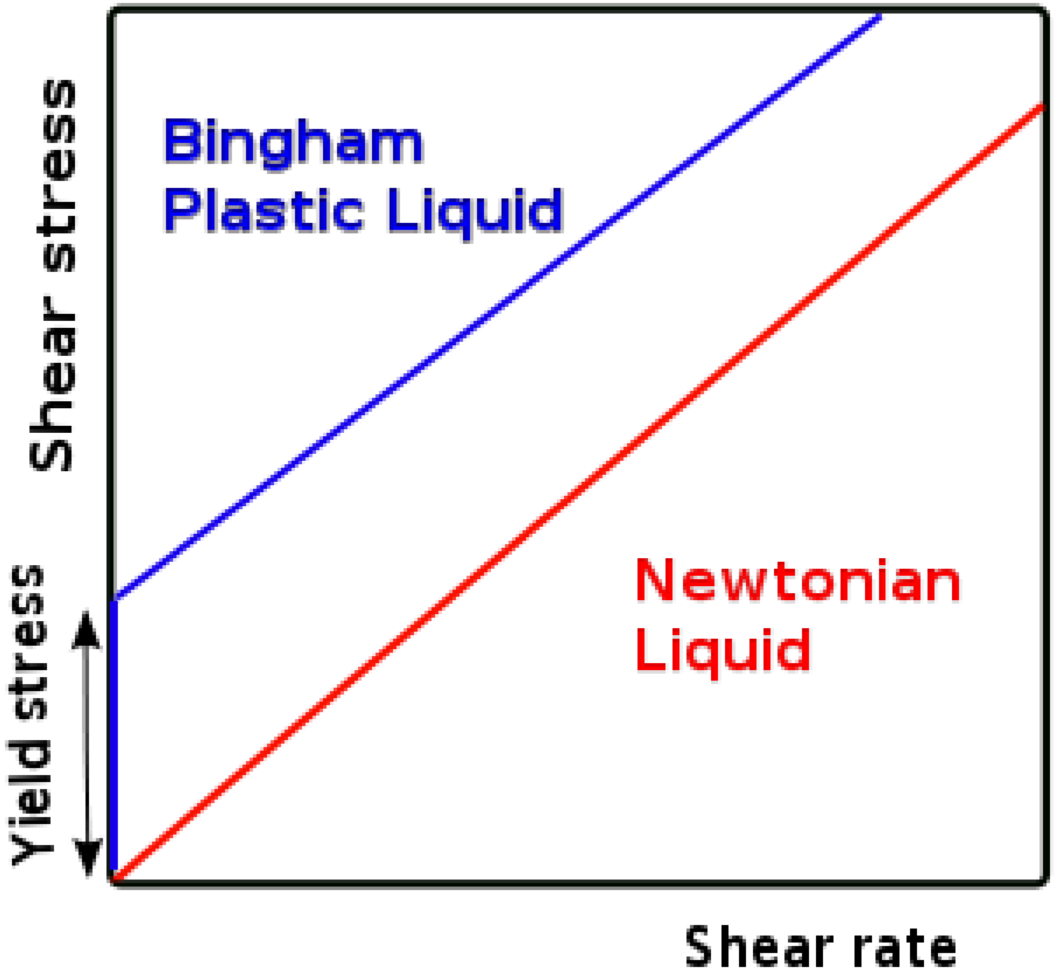
- **Dimensionless form**

$$\tau = \left[1 + \frac{B_i}{\dot{\gamma}}\right]\dot{\gamma} \quad \Longleftrightarrow \quad \tau > B_i$$

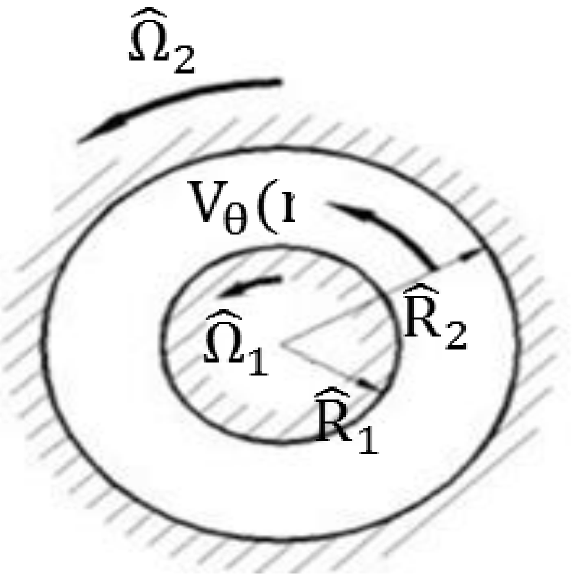
$$\dot{\gamma} = 0 \quad \Longleftrightarrow \quad \tau \le B_i$$

$$B_i = \frac{\hat{\tau}_y}{\hat{\mu}_{ref}\,\hat{R}_1\hat{\Omega}_1/\hat{d}}$$

Reference viscosity $\boxed{\hat{\mu}_{ref} = \hat{\mu}_p}$



Base flow



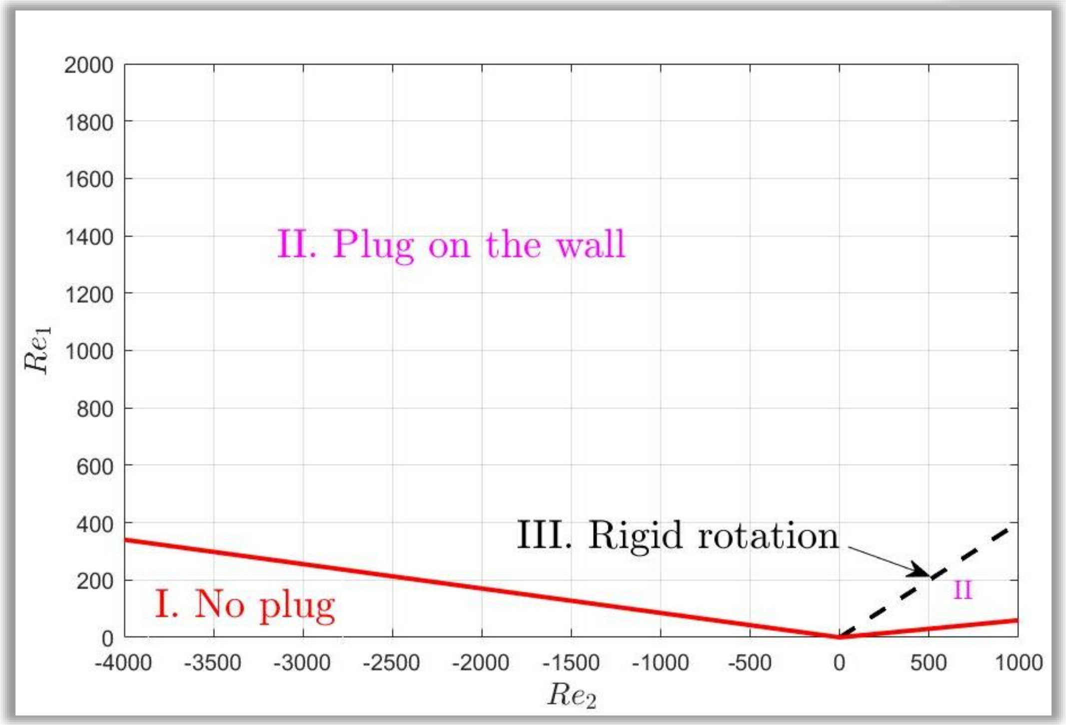
- Purely azimuthal base flow:

$$U_b = (0, V_b, 0)$$

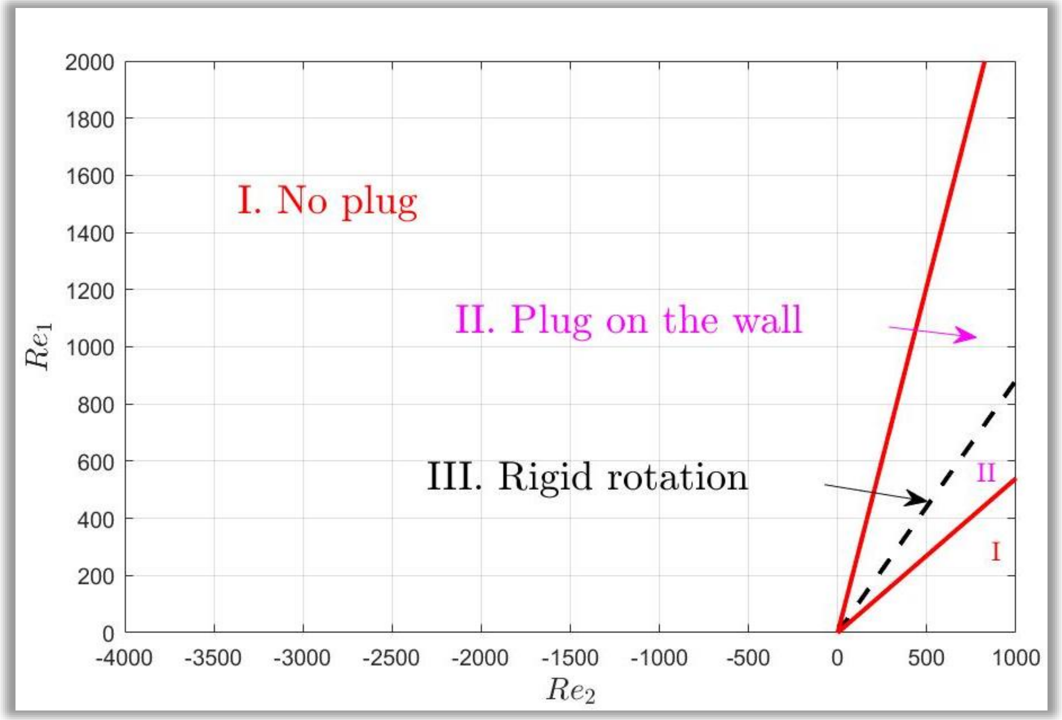
$$\frac{d}{dt}(r^2\tau_{r\theta}) = 0 \qquad V_b(R_1) = 1 \qquad V_b(R_2) = Re_2/Re_1$$

- Different configurations:

$$B = 5$$



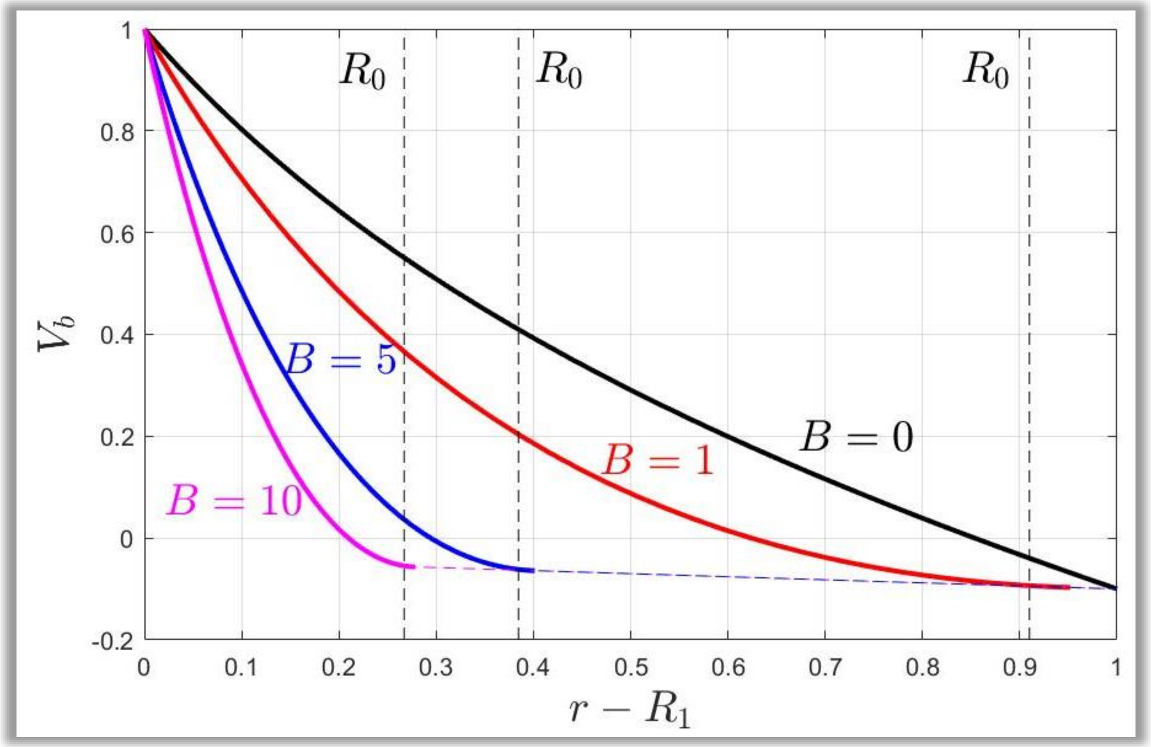
Wide gap ($\eta = 0.4$)



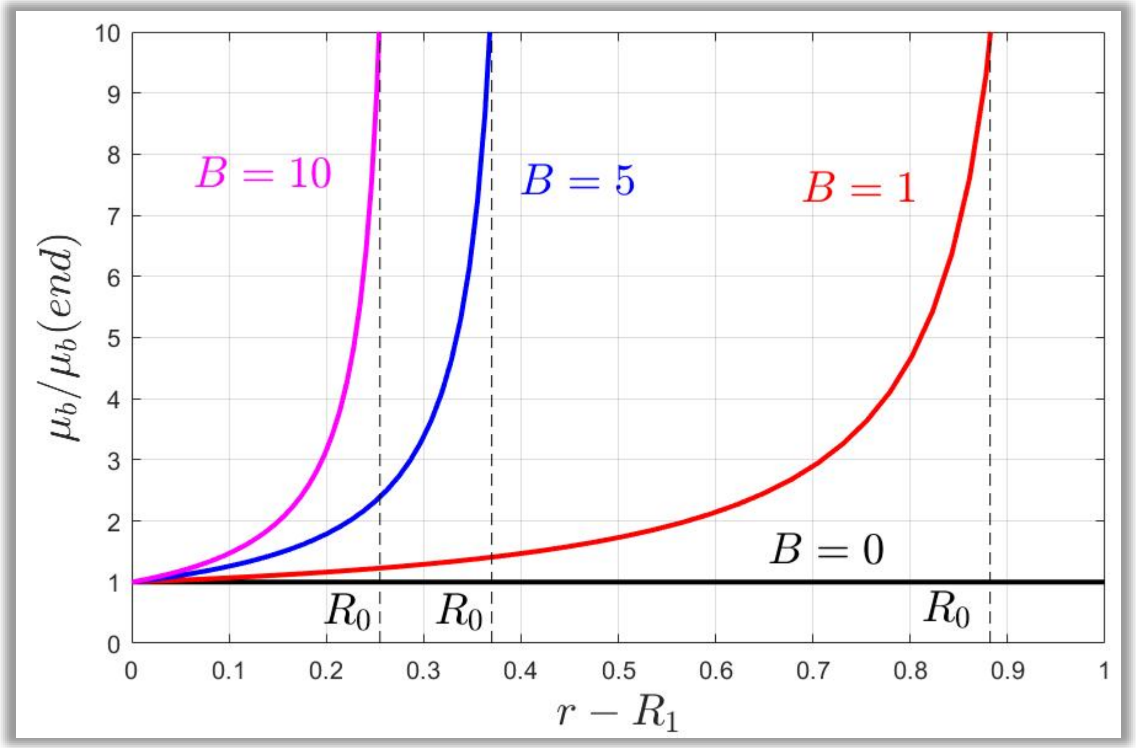
Narrow gap ($\eta = 0.883$)

Influence of Bingham number on velocity and viscosity profiles for wide gap and contra-rotating case

$\eta = 0.4$, $Re_2 = -100$, $Re_1 = 1000$



Azimuthal velocity profiles

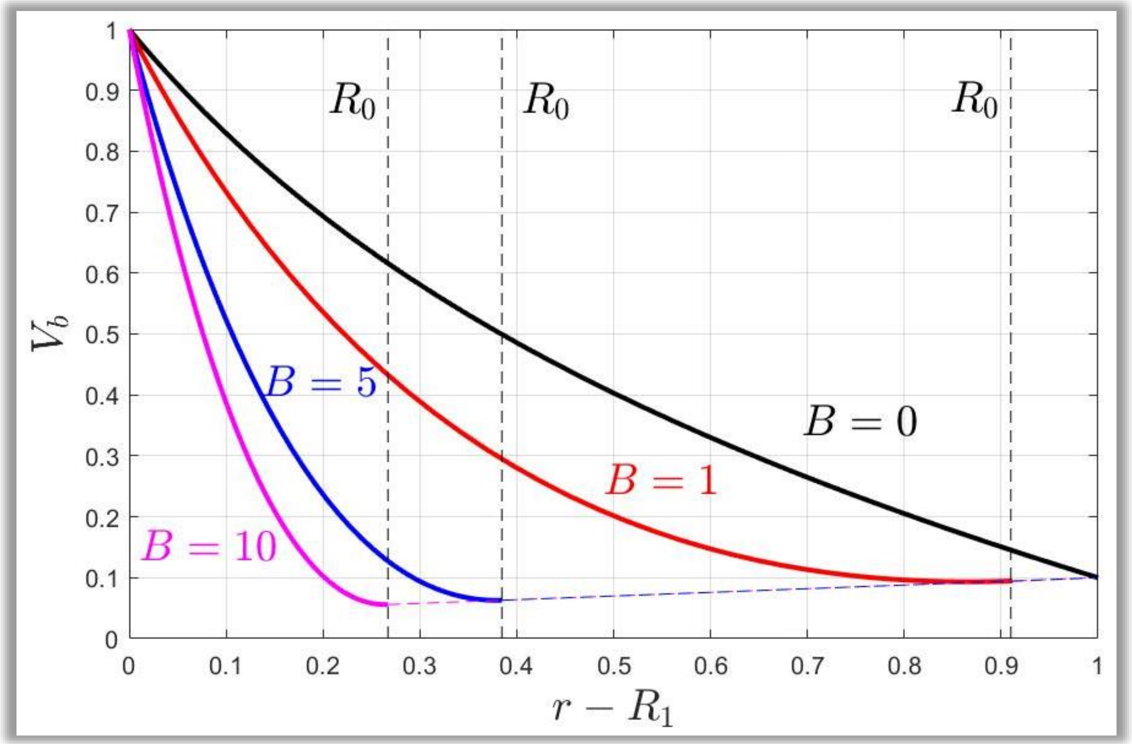


Viscosity profiles

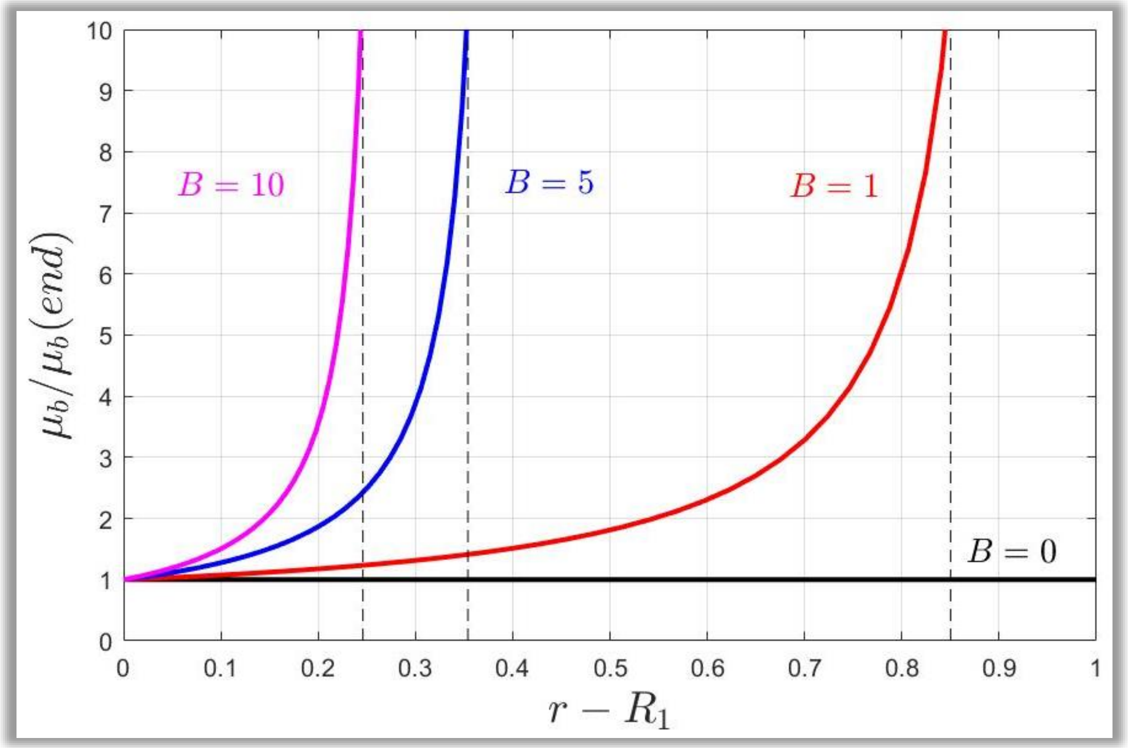
It is important to see that with increasing of Bingham number, the region where we have flow is reduced, but also the velocity gradient increases. In viscosity profile, near the yield surface, the viscosity tends to infinity

Influence of Bingham number on velocity and viscosity profiles for wide gap and co-rotating case

$$\eta = 0.4 \quad , \quad Re_2 = 100 \quad , \quad Re_1 = 1000$$



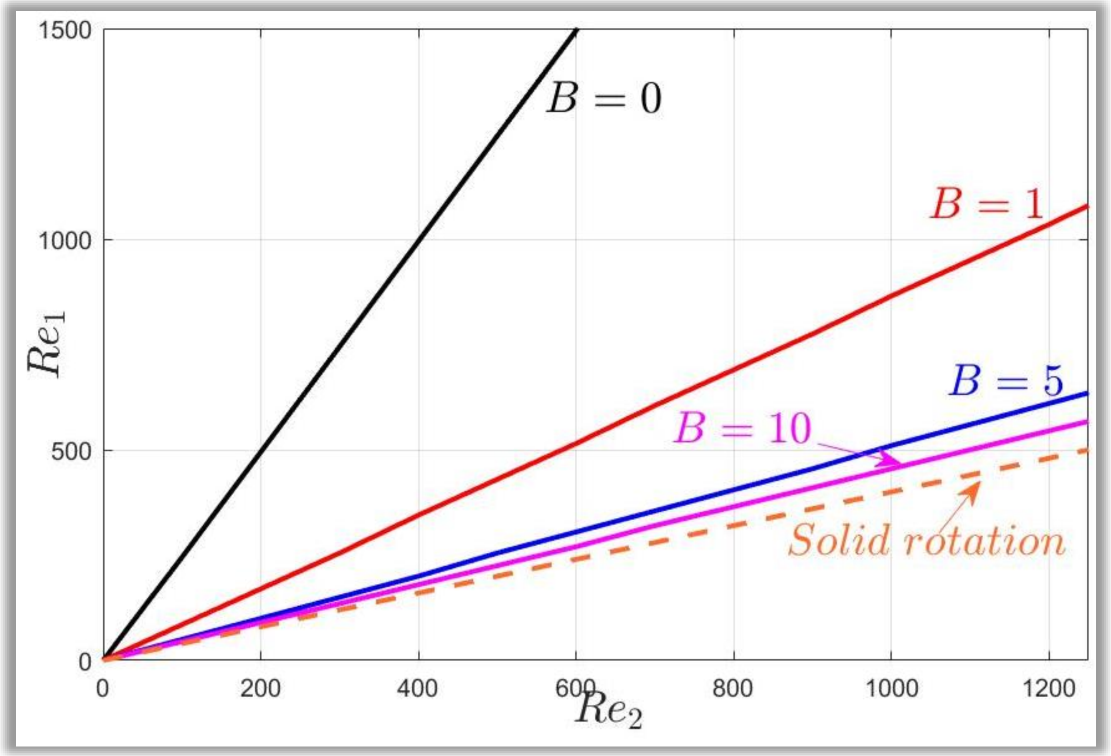
Azimuthal velocity profiles



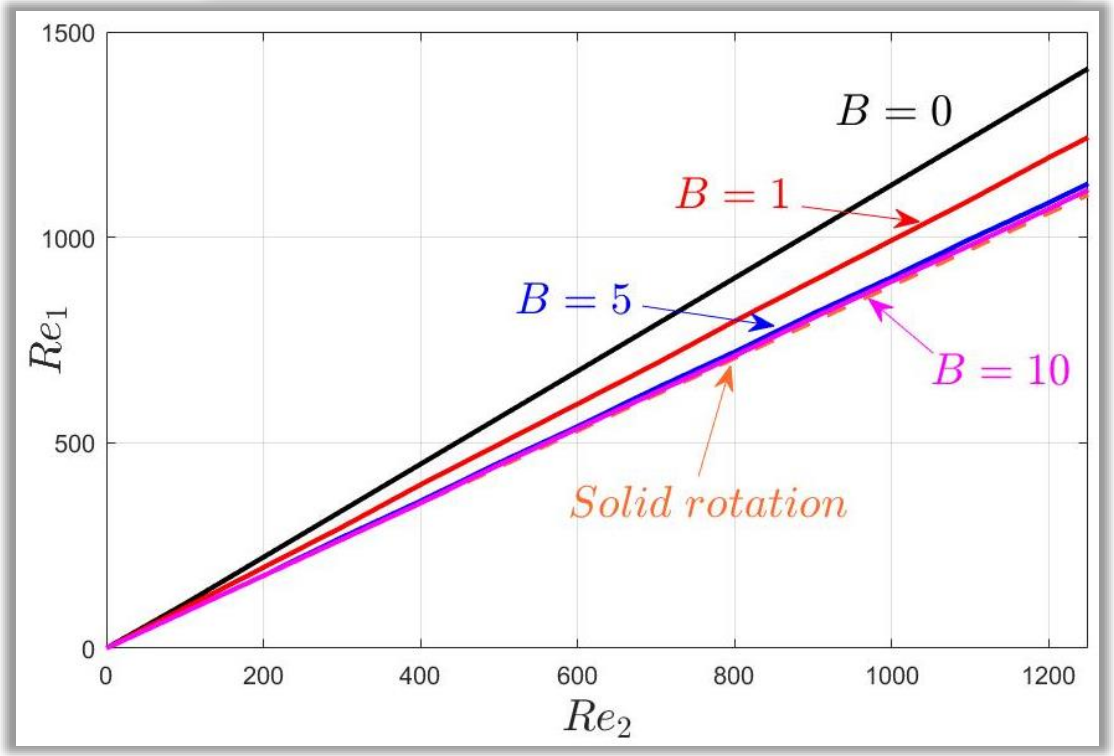
Viscosity profiles

Rayleigh line

Rayleigh stability criterion for a perfect fluid: $\dfrac{d(rV_b)^2}{dr} < 0 \ ,\forall\, r = [R_1, R_2]$



Wide gap (η = 0.4)



Narrow gap (η = 0.883)

The increasing of the Bingham number brings the Rayleigh line closer to the solid rotation line

Linear stability analysis

$$\{\boldsymbol{U}, P, \tau\} = \{\boldsymbol{U_b}, P_b, \tau_b\} + \{\boldsymbol{u}', p', \tau'\}$$

- Linearized perturbation equations:

$$\nabla \cdot \boldsymbol{u}' = 0$$

$$\frac{\partial \boldsymbol{u}'}{\partial t} + Re_1\left[(\boldsymbol{u}' \cdot \nabla)\boldsymbol{U_b}, +(\boldsymbol{U_b} \cdot \nabla)\boldsymbol{u}'\right] = -\nabla p' + \nabla \tau'$$

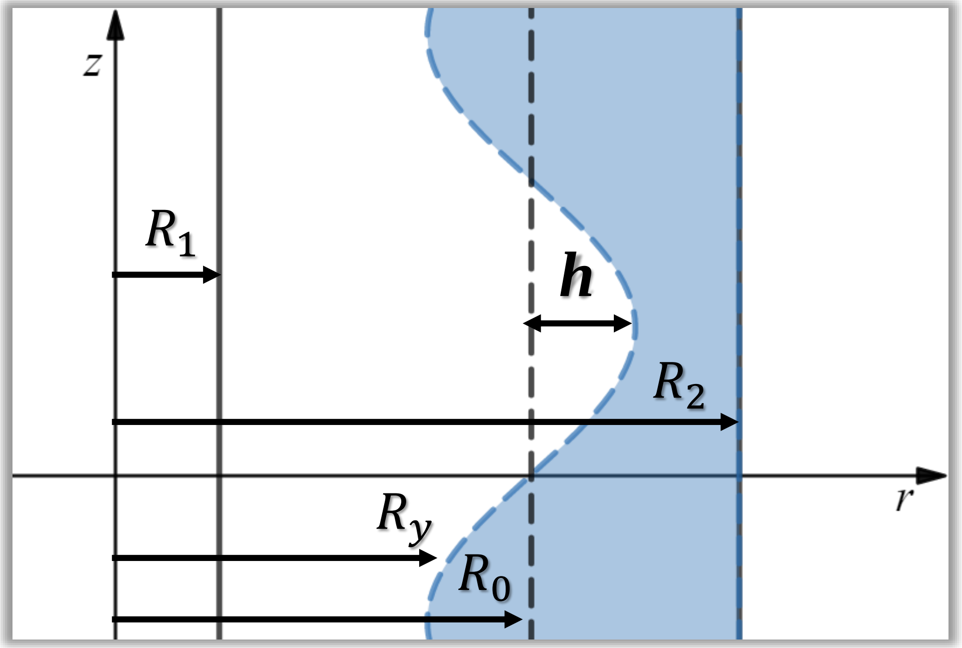
- Perturbation of the shear-stress :

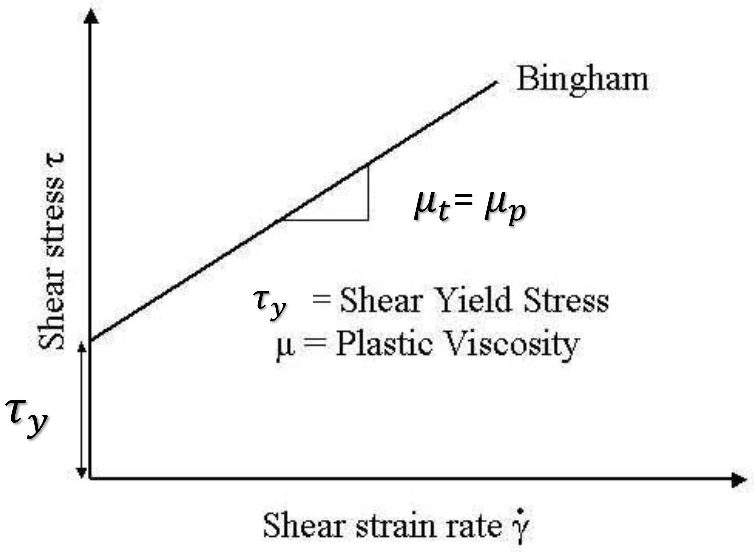
$$\tau'_{ij} = [\tau_{ij}(\boldsymbol{U_b} + \boldsymbol{u}') - \tau_{ij}(\boldsymbol{U_b})] = \mu_b \dot{\gamma}_{ij}(\boldsymbol{u}') + (\mu_t - \mu_b)A$$

- Tangent viscosity: $\mu_t = [\partial\tau_{r\theta}/\partial\dot{\gamma}_{r\theta}]_b = 1$

$$A_{i,j} = 0 \quad if \quad i,j \neq r\theta, \theta r$$

$$A_{r\theta} = A_{\theta r} = \dot{\gamma}_{r\theta}(\boldsymbol{u}')$$





Problem with eigenvalues

❖ Solution in the form of normal modes:

$$\{u', v', w', p'\} = [u(r), v(r), w(r), p(r)] \exp[i(m\theta + kz)] \exp[\sigma t]$$

$m \in \mathbb{N}$: azimuthal wavenumber
$k \in \mathbb{R}$: axial wavenumber
$\sigma = \sigma_r + i\sigma_i$ – complex eigenvalue;
σ_i – principal of stability exchange and σ_r – amplification of the perturbation

❖ Problem with generalized eigenvalues:

$$\sigma M X = \mathcal{L} X \quad , \qquad X = (u, v)^T$$

where, $\mathcal{L}$ and M – linear operators

The resolution of the eigenvalue problem is done using a spectral discretization based on the Chebyshev collocation method and eigenvalue problem is solved directly using Mathlab

Boundary conditions

❖ Annular space is fully sheared : region I

- No – slip velocity at the wall

$$u' = v' = 0 \quad at \quad r = R_1, R_2$$

$$\underbrace{Du' = 0}_{\nabla \cdot u' = 0} \quad at \quad r = R_1, R_2$$

❖ Annular space is partially sheared : region II

$$u' = v' = 0 \;\; at \quad r = R_1$$

$$u' = v' = 0 \;\; at \quad r = R_0 + h$$

- Compatibility conditions :

$$\dot{\gamma}_{ij}(\boldsymbol{U_b} + \boldsymbol{u'}) = 0 \quad , \qquad r = R_0 + h$$

h : perturbation of the interface "liquid – solid"

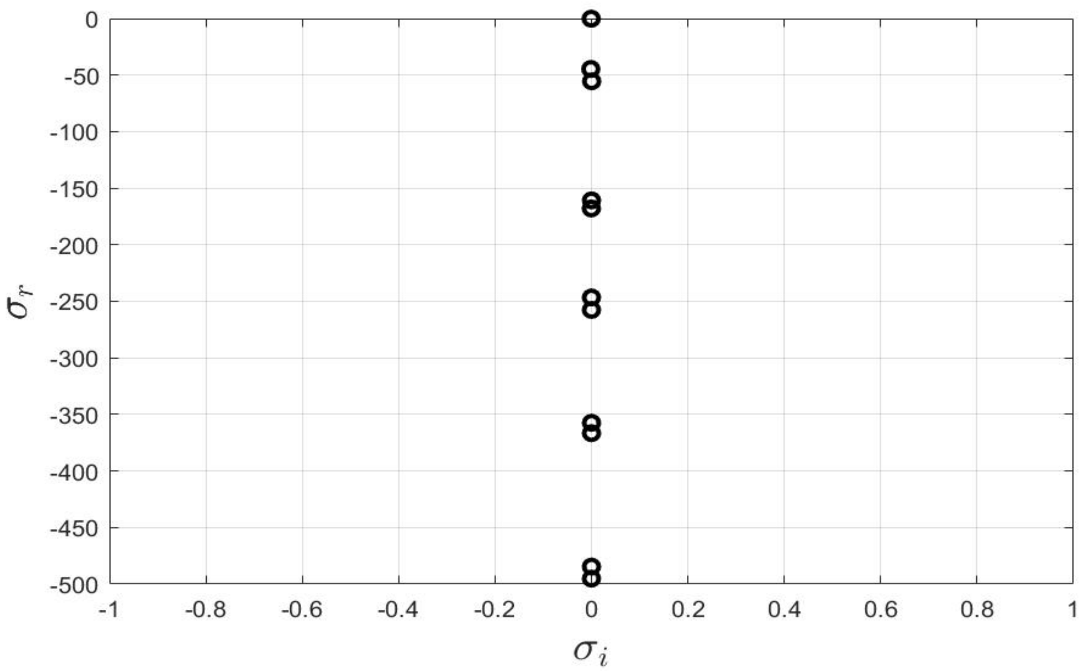
At the linear order:

$$u' = Du' = v' = 0 \quad at \quad r = R_o$$

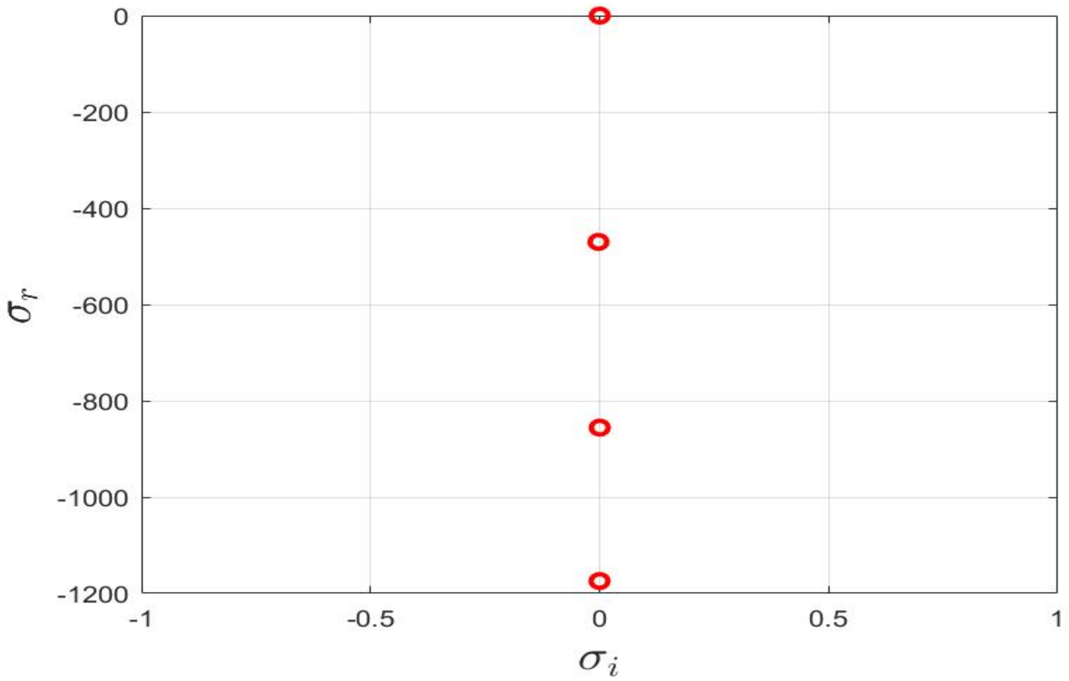
$$Dv' + hD^2V_b = 0 \quad at \quad r = R_o$$

❖ Eigenspectra for perturbations

$\eta = 0.4$



$B = 0$

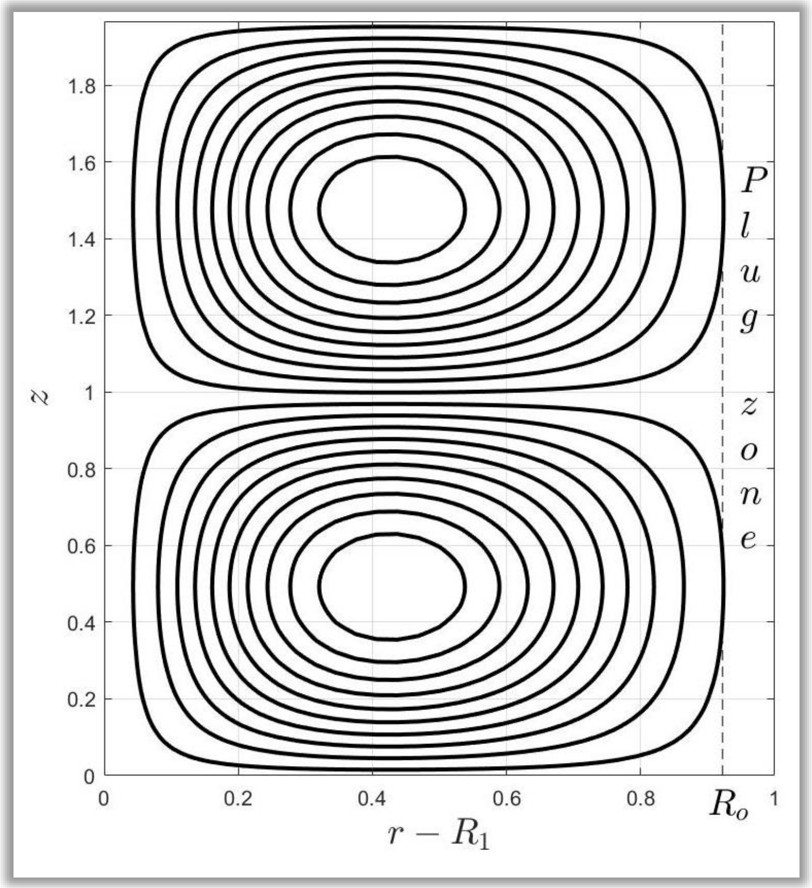


$B = 5$

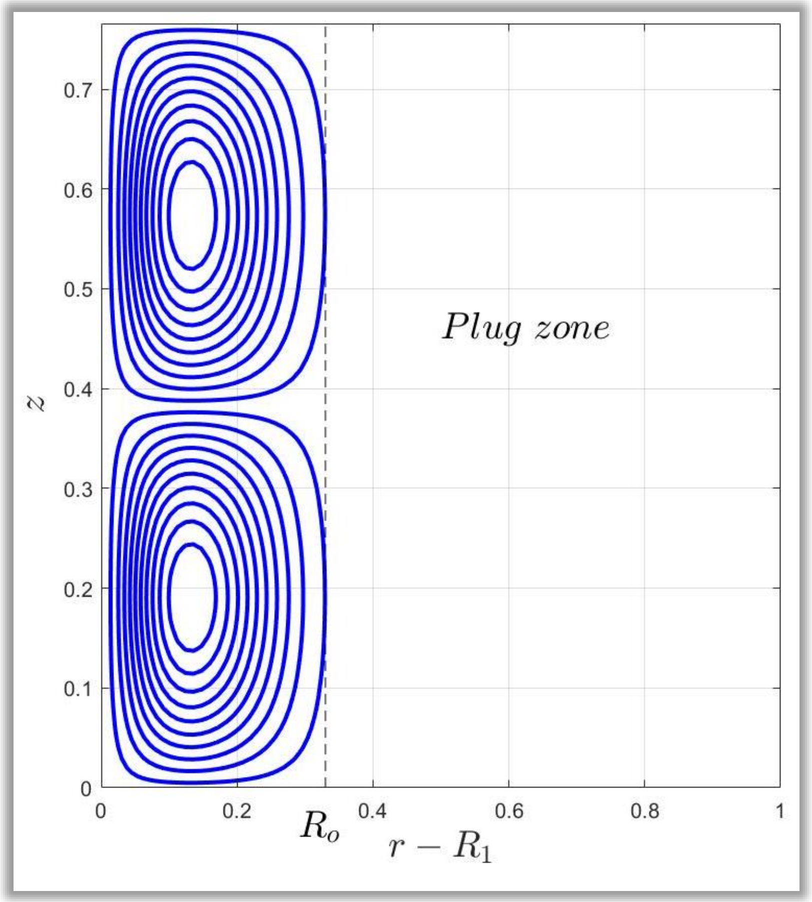
The peculiarity of Bingham effects is emphasized by the spectra of eigenvalues. For axisymmetric perturbations the eigenvalues are real or complex conjugate. We can mention that with increasing Bingham number, separation between two eigenvalues becomes more important.

❖ Contours of the radial velocity

$\eta = 0.4$



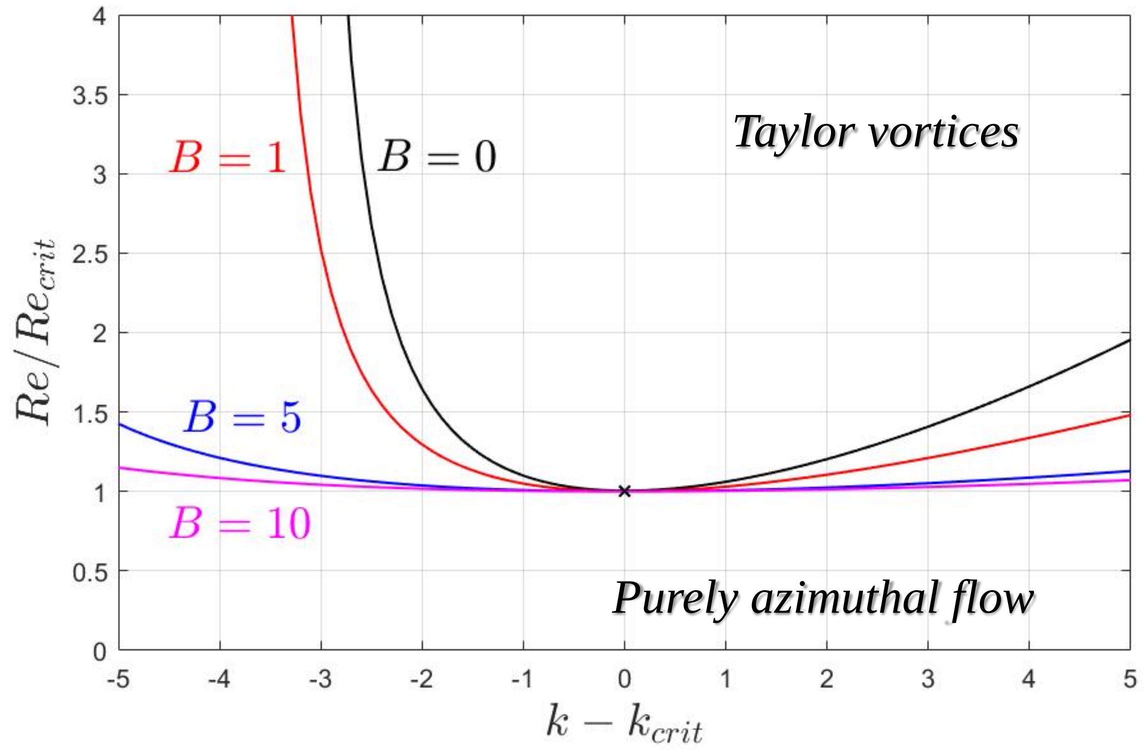
$B = 0$



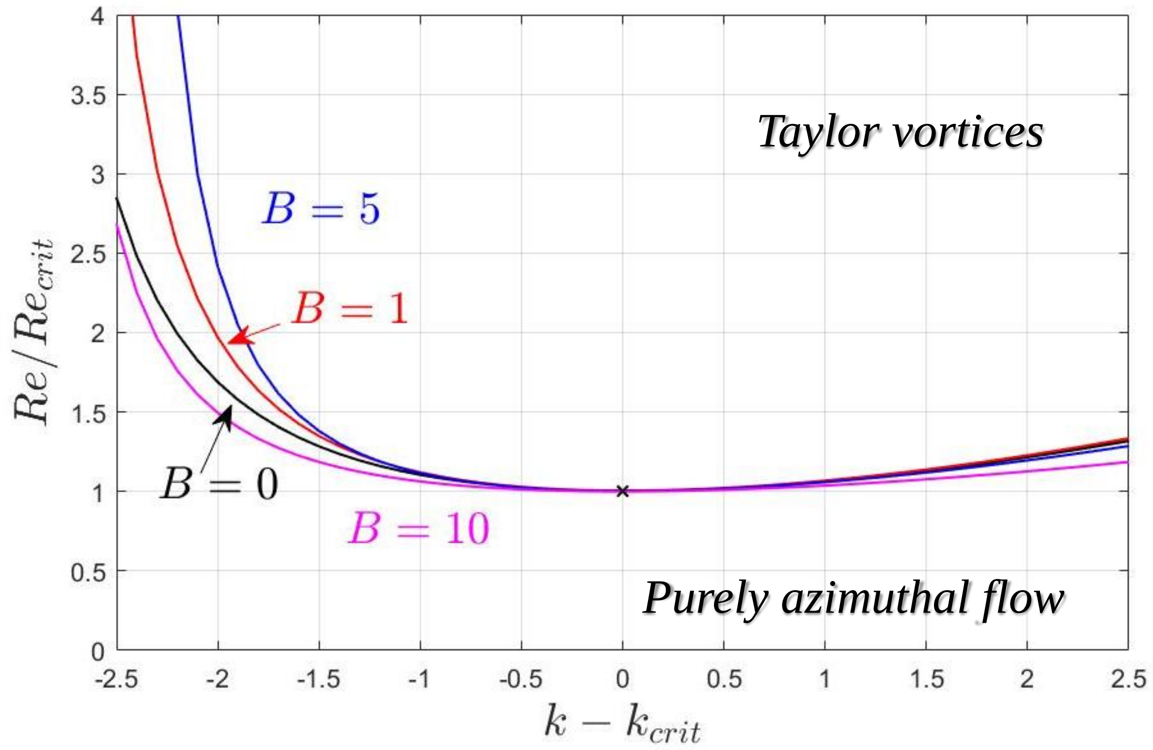
$B = 5$

We can notice that with increasing the Bingham number vortices are squeezed towards the inner wall

Marginal stability curves



Wide gap ($\eta = 0.4$)

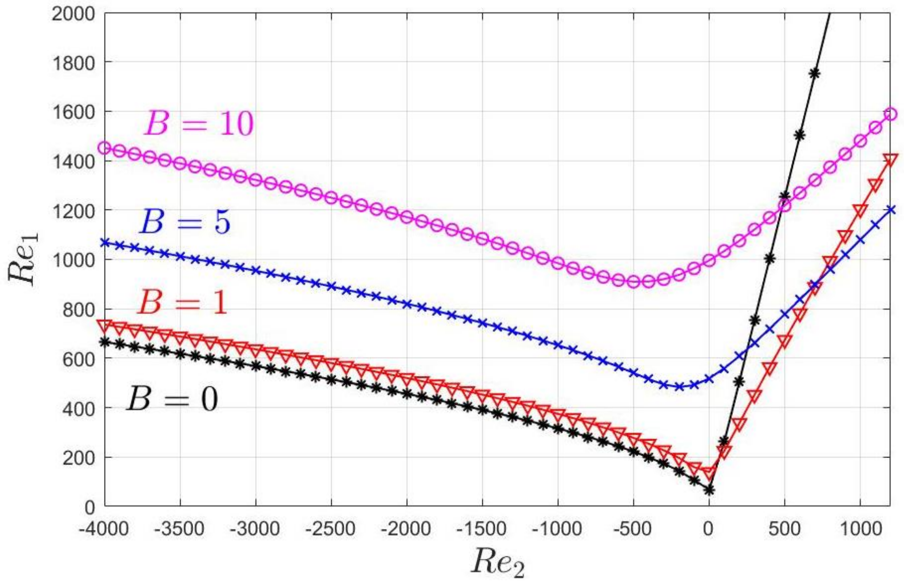


Narrow gap ($\eta = 0.883$)

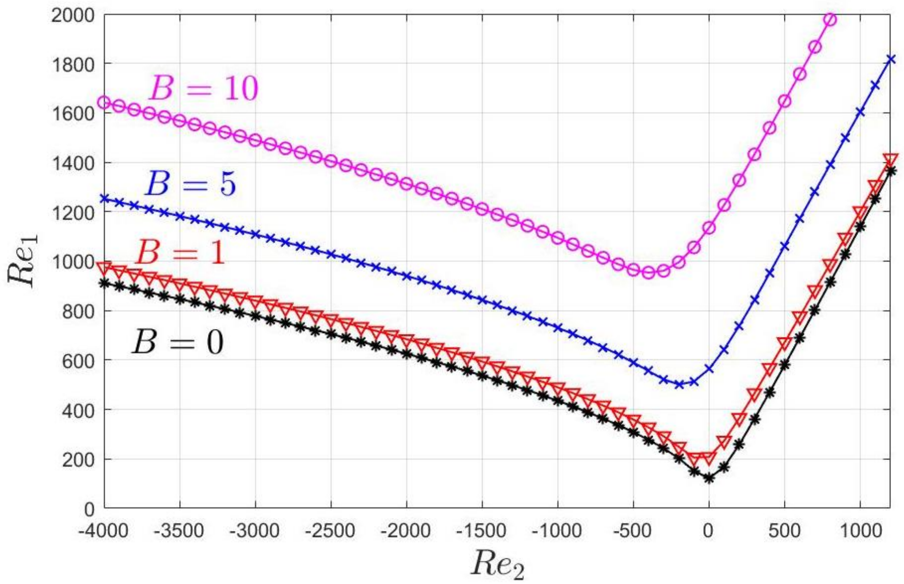
With increasing Bingham number marginal stability curves are flatten. The marginal stability curve is a boundary between Taylor vortices and purely azimuthal flow.

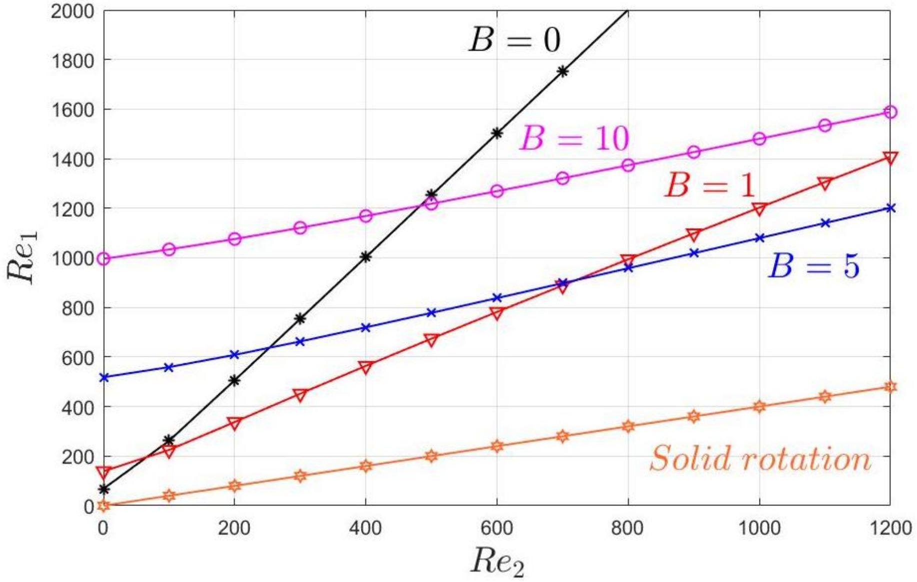
Critical conditions

Wide gap ($\eta = 0.4$)



Narrow gap ($\eta = 0.883$)

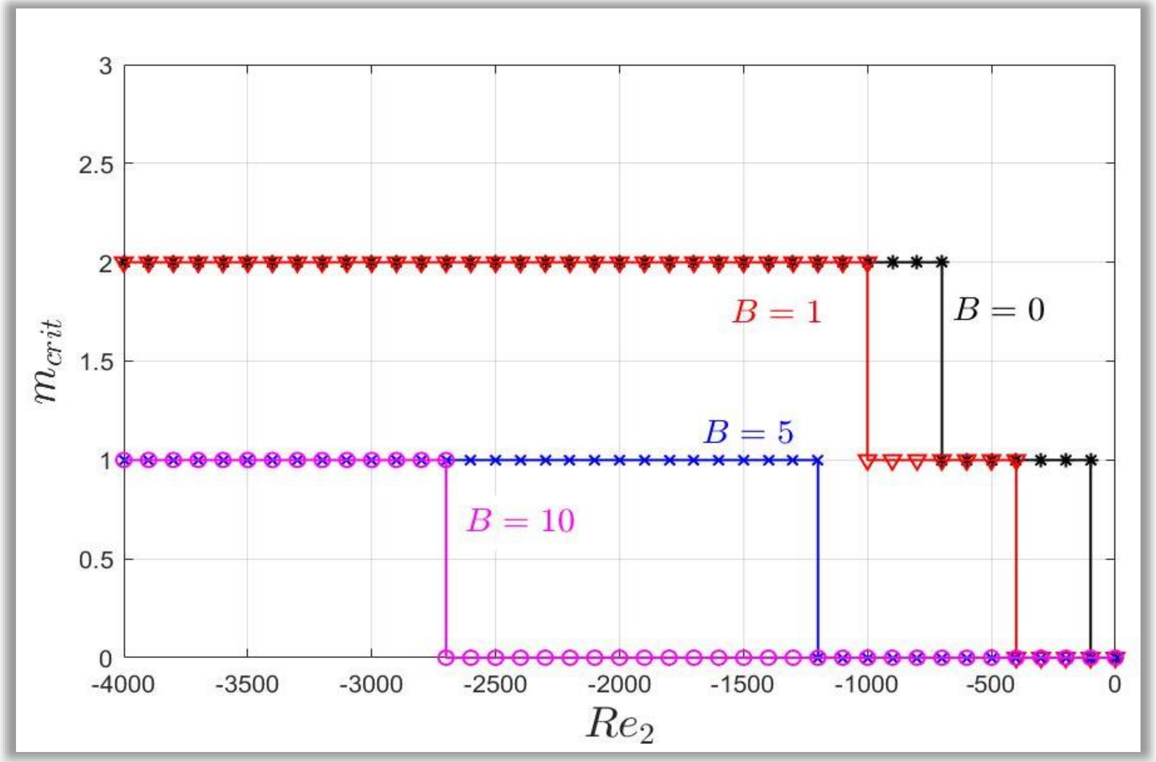




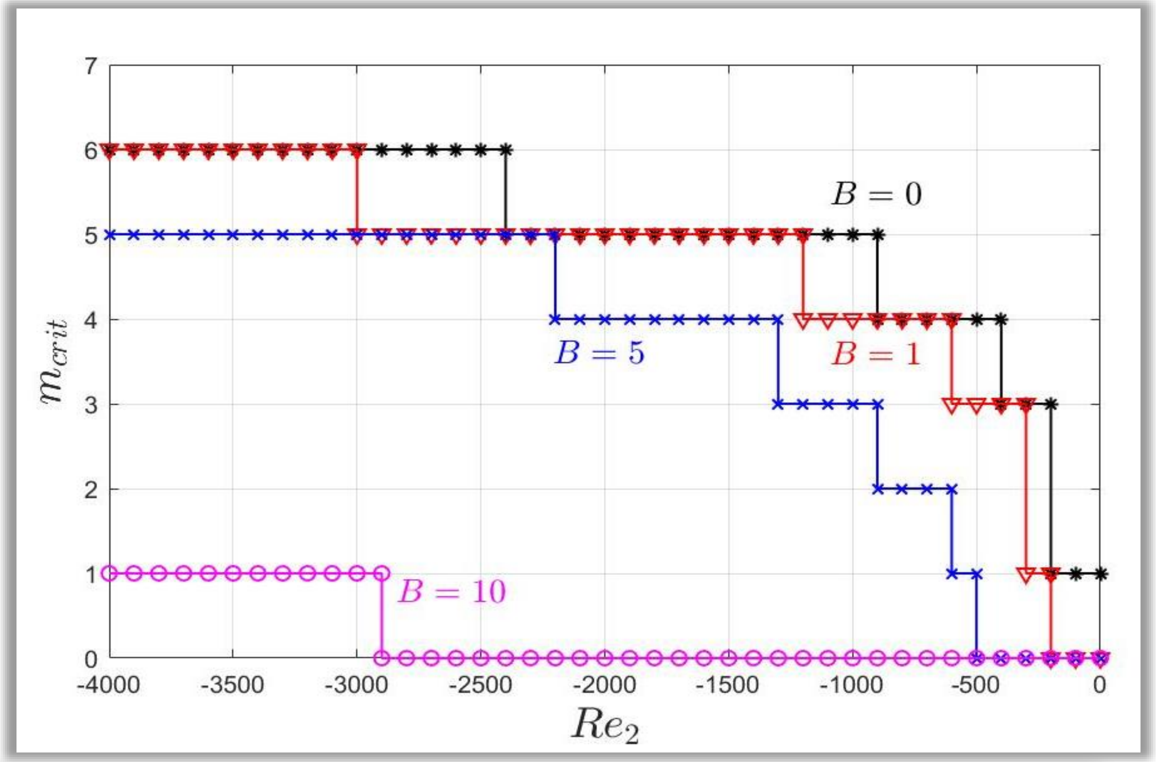
- Low Re_1 variation with Re_2 for high Bingham values
- Stabilizing effect of Bingham's number in narrow gap
- Destabilizing effect of Bingham number in a wide gap for co-rotational case: The increase of the wall velocity gradient outweighs the viscous dissipation

Critical conditions

Wide gap ($\eta = 0.4$)



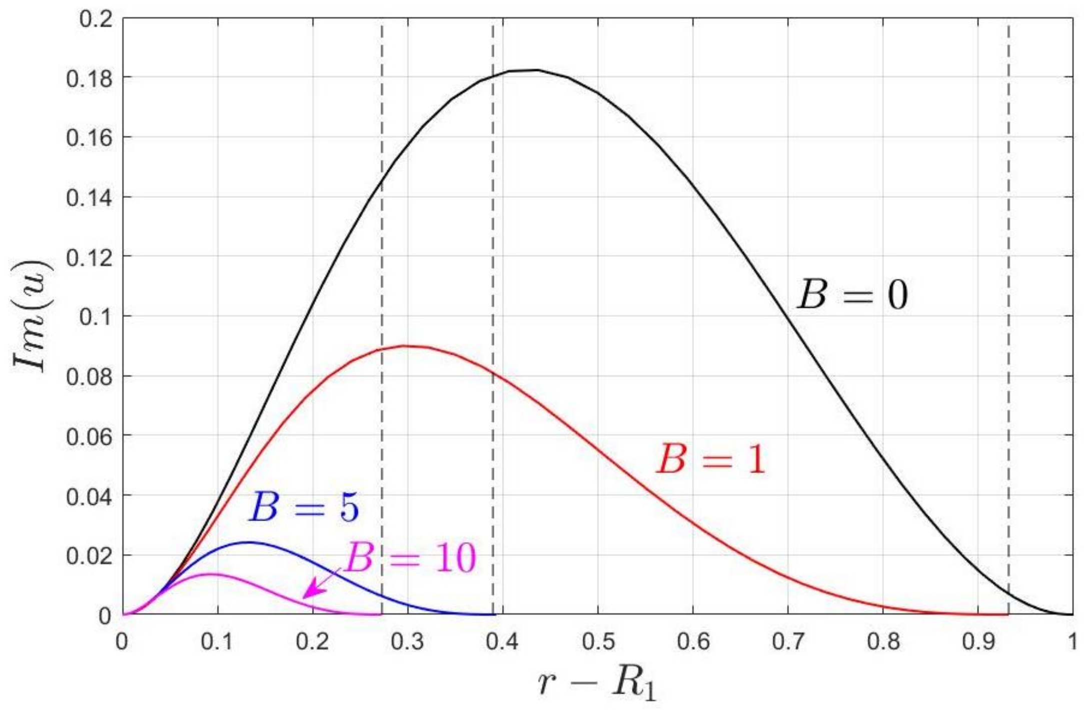
Narrow gap ($\eta = 0.883$)

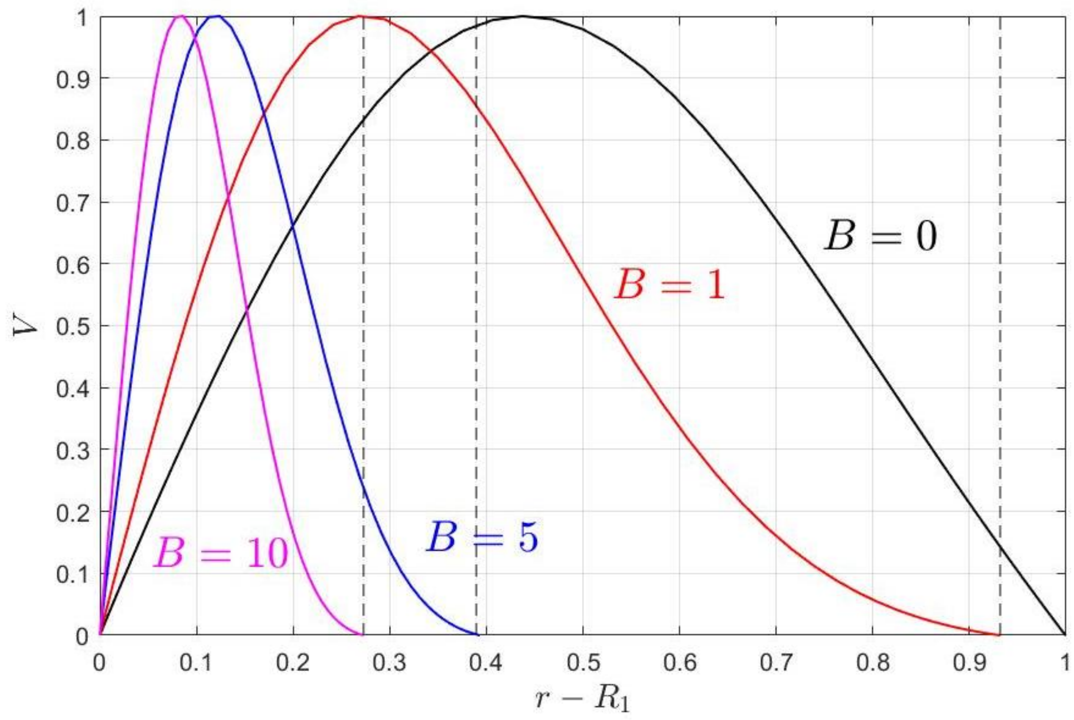


For a high value of Bingham number, the critical mode remains axisymmetric over a wide range of Re_2. The size of the structures decreases with the increase of Re_2.

Eigenfunctions

Wide gap ($\eta = 0.4$)

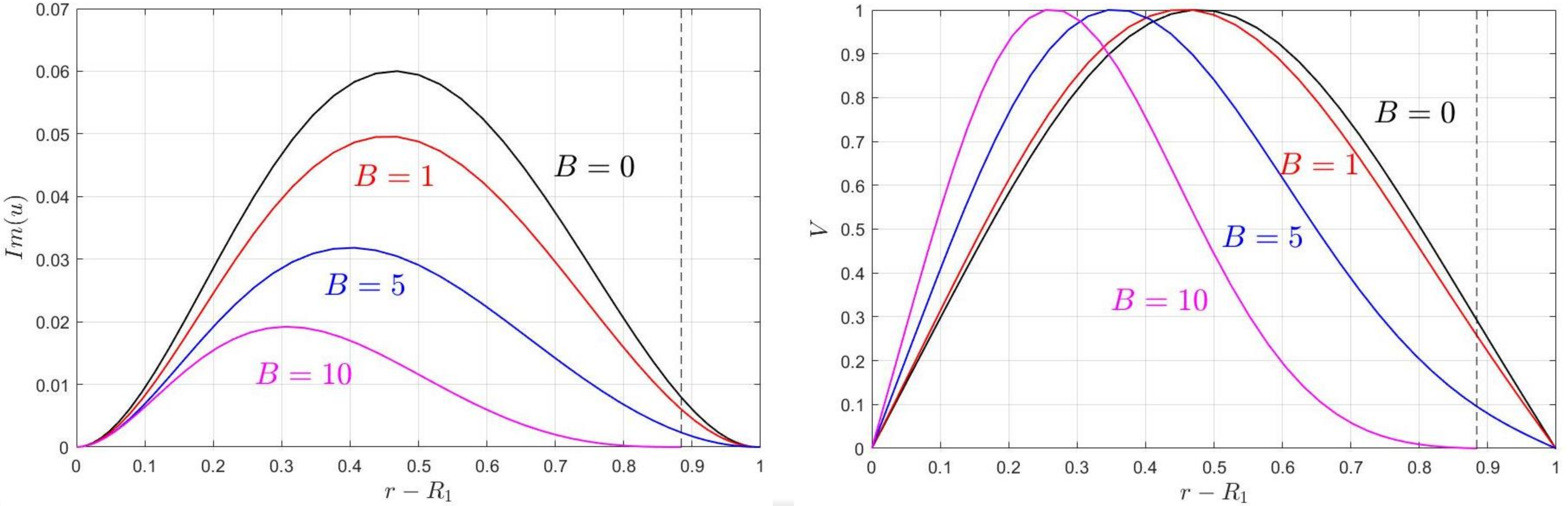




Radial velocity decreases with increasing Bingham number for wide gap and squeezed towards inner wall. The azimuthal velocity is squeezed to the inner wall.

Eigenfunctions

Narrow gap ($\eta = 0.883$)



The radial velocity deceases with increasing Bingham number for narrow gap and squeezed towards inner wall. The azimuthal velocity is squeezed to the inner wall. But the effect is less than for a wide gap.

Weakly nonlinear analysis

- Governing equations and boundary conditions are compactly written as :

$$\frac{\partial}{\partial t} M(\psi) = L(\psi) + N_{\mathrm{I}}(\psi, \psi) + N_{\mathrm{I}}(\psi, \psi, \psi, \dots) \quad , \qquad \psi = (u, v)$$

Linear terms

Nonlinear inertial terms

Nonlinear viscous terms

- Using Fourier series in the axial direction and amplitude expansion method :

$$\psi(r, z, t) = \sum_{n=-\infty}^{+\infty} \psi_n(r, t) \exp[inkz]$$

$$\psi_n(r, t) = \sum_{m=0}^{N} \psi_{n,2m+n}(r) A^n |A|^{2m}$$

- Amplitude equation :

$$\frac{\partial A}{\partial t} = \sum_{m=0}^{+\infty} g_m |A|^{2m}$$

Stuard –Landau equations

- Perturbation of the yield surface:

$$h = \sum_{n=1} \sum_{m=0} h_{n,2m+n} |A|^{2m} A^n E^n + \sum_{m=1} h_{0,2m} |A|^{2m}$$

❖ First order :

Linear problem (fundamental mode)

$$\boxed{\psi \sim A\,\psi_{11}\exp[ikz] + c.c.}$$

❖ Second order :

- Second harmonic: Interaction of the fundamental mode with itself

$$\boxed{\psi \sim A^2\,\psi_{22}\exp[2ikz] + c.c.}$$

- Modification of the base flow : Interaction of the fundamental mode with its complex conjugate

$$\boxed{\psi \sim |A|^2\,\psi_{02} + c.c.}$$

Boundary conditions

No – slip velocity

$$\boldsymbol{u}'(R_1) = 0 \quad , \qquad \boldsymbol{u}'(R_0 + h) = 0$$

❖ First order :

$$U_{11} = DU_{11} = V_{11} = 0$$

❖ Second order :

$$U_{22} = DU_{22} = 0$$

$$V_{22} + h_{11}DV_{11} + \frac{h_{11}^2}{2}D^2V_b = 0$$

$$U_{02} = DU_{02} = 0$$

$$V_{02} + h_{11}DV_{11}^* + h_{11}^*DV_{11} + \frac{|h_{11}|^2}{2}D^2V_b = 0$$

Compatibility conditions

According to the Bingham model :

$$\dot{\gamma}_{ij}(\boldsymbol{U_b} + \boldsymbol{u}')\Big|_r = 0 \quad , \qquad r = R_0 + h$$

- First order :

$$DV_{11} + h_{11} D^2 V_b = 0$$

- Second order :

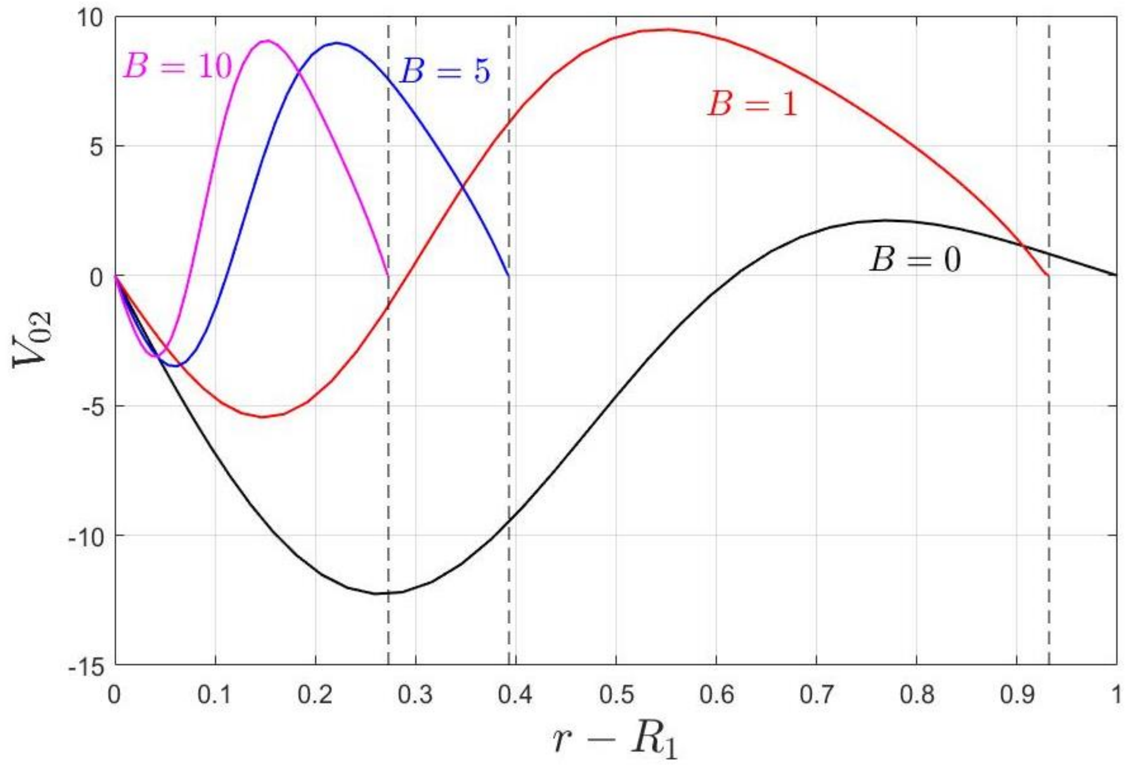
$$D^2 U_{22} + h_{11} D^3 U_{11} = 0$$

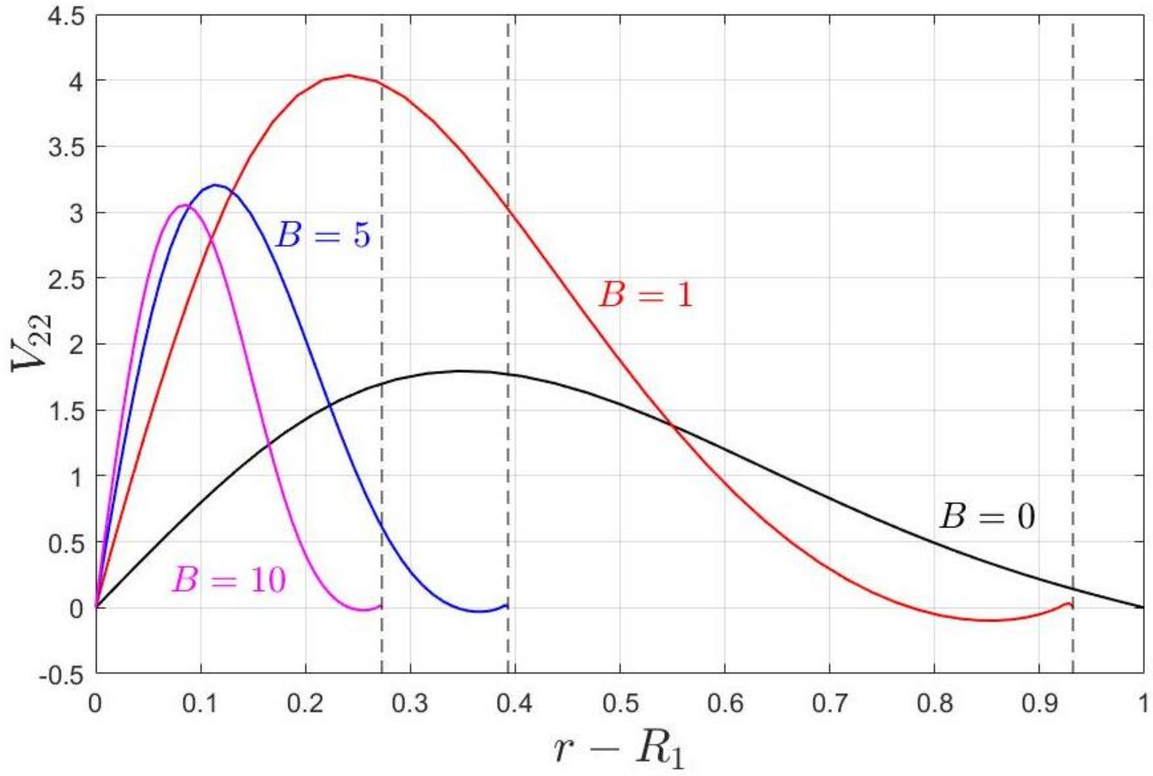
$$V_{22} + \frac{1}{2} h_{11} D V_{11} = 0$$

$$DV_{22} + \frac{1}{2\,r} h_{11} D V_{11} + h_{11} D^2 V_{11} + h_{11} D^2 V_b + \frac{1}{2}(h_{11})^2 \left(D^3 V_b - \frac{1}{r} D^2 V_b \right) = 0$$

$$DV_{02} + h_{11} D^3 V_b + h_{02} D^2 V_b + h_{11} D^2 V_{11}^* + h_{11}^* D^2 V_{11} = 0$$

The second order problem : $\eta = 0.4$





Modification of the base flow: asymmetry between the inner wall and outer wall.
Second harmonic: it is radiused and it is completely different from linear problem, where perturbation are very small.

Conclusion

- **Base flow**

  Different configurations of the base flow as a function of the Bingham number

- **Linear stability analysis**

  - The critical Rayleigh number increases with increasing Bingham number
  - One situation where the critical Reynolds number decreases with increasing Bi
  - The marginal stability curve flattens with increasing Bi

- **Weakly nonlinear stability analysis**

  - Second harmonic is significant comparatively to the eigenfunction
  - Focus on the boundary conditions and compatibility conditions

Future

Determination of the first Landau-constant: nature of the primary bifurcation

Thank you for your attention